Axis-Angle representation

Use an axis an angle to represent an orientation

ϕ = angle rotate about $\bar{r} = (r_x, r_y, r_z)^T = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}$

$r_x^2 + r_y^2 + r_z^2 = 1$ ⟵ unit vector

$$\bar{R}_{\bar{r},\phi} = \begin{bmatrix} r_x^2 V\phi + C\phi & r_x r_y V\phi - r_z S\phi & r_x r_z V\phi + r_y S\phi \\ r_x r_y V\phi + r_z S\phi & r_y^2 V\phi + C\phi & r_y r_z V\phi - r_x S\phi \\ r_x r_z V\phi - r_y S\phi & r_y r_z V\phi + r_x S\phi & r_z^2 V\phi + C\phi \end{bmatrix}$$

$S\phi = \sin\phi$

$C\phi = \cos\phi$

$V\phi = \text{vers}\,\phi = 1 - \cos\phi$

$$\text{Trace}(\bar{R}_{\bar{r},\phi}) = r_{11} + r_{22} + r_{33} = Tr(\bar{R})$$

$$\begin{aligned} Tr(\bar{R}_{\bar{r},\phi}) &= r_x^2 V\phi + C\phi + r_y^2 V\phi + C\phi + r_z^2 V\phi + C\phi \\ &= (r_x^2 + r_y^2 + r_z^2) V\phi + 3C\phi \\ &= V\phi + 3C\phi \\ &= 1 - C\phi + 3C\phi \\ Tr(\bar{R}) &= 1 + 2\cos\phi \end{aligned}$$

$$\Rightarrow \boxed{\phi = \pm\cos^{-1}\left(\frac{Tr(\bar{R}) - 1}{2}\right)}$$

	ϕ?
$Tr(\bar{R}) = 3$ ⇒	0
$Tr(\bar{R}) = 1$ ⇒	±90
$Tr(\bar{R}) = -1$ ⇒	180

$$r_{11} = r_x^2 V\phi + C\phi \quad \Rightarrow \quad r_x = \pm\sqrt{\frac{r_{11} - C\phi}{V\phi}}$$

$$r_{22} = r_y^2 V\phi + C\phi \quad \Rightarrow \quad r_y = \pm\sqrt{\frac{r_{22} - C\phi}{V\phi}}$$

$$r_{33} = r_z^2 V\phi + C\phi \quad \Rightarrow \quad r_z = \pm\sqrt{\frac{r_{33} - C\phi}{V\phi}}$$

$$\bar{R}_{\bar{r},\phi} = \begin{bmatrix} r_x^2 V\phi + c\phi & r_x r_y V\phi - r_z s\phi & r_x r_z V\phi + r_y s\phi \\ r_x r_y V\phi + r_z s\phi & r_y^2 V\phi + c\phi & r_y r_z V\phi - r_x s\phi \\ r_x r_z V\phi - r_y s\phi & r_y r_z V\phi + r_x s\phi & r_z^2 V\phi + c\phi \end{bmatrix}$$

$$r_{21} - r_{12} = 2 r_z s\phi \quad \Rightarrow \quad r_z = \frac{r_{21} - r_{12}}{2 s\phi}$$

$$r_{13} - r_{31} = 2 r_y s\phi \quad \Rightarrow \quad r_y = \frac{r_{13} - r_{31}}{2 s\phi}$$

$$r_{32} - r_{23} = 2 r_x s\phi \quad \Rightarrow \quad r_x = \frac{r_{32} - r_{23}}{2 s\phi}$$

But what if $s\phi = 0$?

$\phi = 0$ or 180 — axis is arbitrary

$$s\phi = 0 \quad c\phi = -1 \quad V\phi = 2$$

$$\begin{bmatrix} 2r_x^2 - 1 & 2r_x r_y & 2r_x r_z \\ 2r_x r_y & 2r_y^2 - 1 & 2r_y r_z \\ 2r_y r_z & 2r_y r_z & 2r_z^2 - 1 \end{bmatrix}$$

$$r_{11} = 2r_x^2 - 1 \qquad 2r_x^2 = r_{11} + 1 \qquad r_x = \pm\sqrt{\frac{r_{11} + 1}{2}}$$

$$r_{12} = 2r_x r_y \qquad r_y = \frac{r_{12}}{2r_x}$$

$$r_{13} = 2r_x r_z \qquad r_z = \frac{r_{13}}{2r_x}$$

OR

$$r_{22} = 2r_y^2 - 1 \Rightarrow r_x = \frac{r_{12}}{2r_y} \qquad r_z = \frac{r_{23}}{2r_y}$$

OR

$$r_{33} = 2r_z^2 - 1 \Rightarrow r_x = \frac{r_{13}}{2r_z} \qquad r_y = \frac{r_{23}}{2r_z}$$