Quaternions: - to describe orientation

Recall $\hat{p}_{xyz} = \bar{R}_{44}\,\hat{P}_{uvw}$ $\equiv$ $\bar{P}_{xyz} = \underset{\uparrow\text{ rotation only}}{\bar{R}_{33}\,\bar{P}_{uvw}} + \underset{\uparrow\text{ translation}}{\bar{t}}$

augmented

can concatenate $\bar{p}_{xyz} = \bar{R}_2(\bar{R}_1 \bar{P}_{uvw} + \bar{t}_1) + \bar{t}_2$

$= \bar{R}_2\bar{R}_1 P_{uvw} + \bar{R}_2\bar{t}_1 + \bar{t}_2$ ← can work this way.

Quaternions handle the rotation part.

$$\bar{R}_{33} \equiv \overset{3}{\text{Euler angles}} \equiv \overset{\text{axis angle}}{Rot(\bar{n}, \theta)} \equiv \mathring{q} = (s, a, b, c)^T = \left(\cos\tfrac{\theta}{2}, \sin\tfrac{\theta}{2} n_i, \sin\tfrac{\theta}{2} n_j, \sin\tfrac{\theta}{2} n_k\right)^T$$

4 dimensional unit vector.

Give the 4 elements units of $\underset{\uparrow\text{ scalar part}}{+1}, \underbrace{\bar{i}, \bar{j}, \bar{k}}$ ← vector part {related to 3 basis vectors of cartesian axes}

↑ 3 imaginary parts.

$\bar{i}^2 = \bar{j}^2 = \bar{k}^2 = -1$

$\bar{i}\bar{j} = \bar{k}$ $\quad \bar{j}\bar{k} = \bar{i}$ $\quad \bar{k}\bar{i} = \bar{j}$ $\quad \bar{j}\bar{i} = -\bar{k}$ $\quad \bar{k}\bar{j} = -\bar{i}$ $\quad \bar{i}\bar{k} = -\bar{j}$

$$\mathring{q} = s + \bar{V} = \underset{\substack{\uparrow \\ \text{scalar part}}}{s} + \underbrace{a\bar{i} + b\bar{j} + c\bar{k}}_{\text{vector part.}}$$

{ includes reals, complex numbers, 3d vectors

Orientation may be represented w/ unit quaternion $s^2 + a^2 + b^2 + c^2 = 1$

Note: Conjugate of $\mathring{q} = \mathring{q}^* = s - \bar{V} = s - a\bar{i} - b\bar{j} - c\bar{k}$ {"inverse of the rotation"}

Product of quaternions $\mathring{q}_1\mathring{q}_2 = (s_1 + a_1\bar{i} + b_1\bar{j} + c_1\bar{k})(s_2 + a_2\bar{i} + b_2\bar{j} + c_2\bar{k}) = (s_1 + V_1)(s_2 + V_2)$

multiply using ordinary algebra distributive law but maintain order and use above relationships ($\bar{i}\bar{j} = \bar{k}$ etc)

$$\text{result} \Rightarrow \underbrace{s_1 s_2 - \bar{V}_1 \cdot \bar{V}_2}_{\text{real part}} + \underbrace{s_2\bar{V}_1 + s_1\bar{V}_2 + \bar{V}_1 \times \bar{V}_2}_{\text{vector part.}}$$

note $\bar{Q}^T \Leftrightarrow \mathring{q}^*$

$$\text{or} \Rightarrow \begin{bmatrix} s_1 & -a_1 & -b_1 & -c_1 \\ a_1 & s_1 & -c_1 & b_1 \\ b_1 & c_1 & s_1 & -a_1 \\ c_1 & -b_1 & a_1 & s_1 \end{bmatrix} \begin{bmatrix} s_2 \\ a_2 \\ b_2 \\ c_2 \end{bmatrix} = \bar{Q}_1 \mathring{q}_2$$

* To concatenate Rotations : multiply the quaternions
  i.e. 16 multiplications vs 27 for R_{33}
* Note, $\mathring{q}$ is easy to "normalize" unlike "R_{33}"
* To find angle between two orientations, θ we know $\cos\frac{\theta}{2} = \mathring{q}_1 \cdot \mathring{q}_2$ (dot product: $s_1 s_2 + \bar{v}_1 \cdot \bar{v}_2$) $=$ real of $\mathring{Q}_1 \mathring{q}_2^*$
  Similar to angle between two planes $\cos\theta = \bar{n}_1 \cdot \bar{n}_2$ (dot product of normals)
* $\mathring{q}$ and $-\mathring{q}$ represent the same rotation otherwise each rotation corresponds to a unique quaternion. All possible orientations are uniformly distributed in 4-d unit sphere ← for Euler angles close orientations may have very different Euler angles
* inverse of a rotation $\mathring{q} = \mathring{q}^*$ (conjugate)
* conversion to axis angle is easy.

To transform a ~~vector~~ point, $\bar{v}$ by rotation $\mathring{q}$, convert $\bar{v}$ to a quaternion $\mathring{\bar{v}} = 0 + \bar{v}$ and multiply $\mathring{q} \cdot \mathring{\bar{v}} \cdot \mathring{q}^*$ (more multiplications than for $R_{33}\bar{v}$)

$\mathring{q} \to R_{33}$

$$\begin{bmatrix} q_0^2 + q_x^2 - q_y^2 - q_z^2 & 2(q_y q_x + q_0 q_z) & 2(q_z q_x + q_0 q_y) \\ 2(q_x q_y + q_0 q_z) & q_0^2 - q_x^2 + q_y^2 - q_z^2 & 2(q_z q_y - q_0 q_x) \\ 2(q_x q_z - q_0 q_y) & 2(q_y q_z + q_0 q_x) & (q_0^2 - q_x^2 - q_y^2 + q_z^2) \end{bmatrix}$$

$s^2 + a^2 - b^2 - c^2$

Note same as Horns definition (opposite minus?)

$R_{33} \to \mathring{q}$

For inverse: Note:

$1 + r_{11} + r_{22} + r_{33} = 4q_0^2$

$1 + r_{11} - r_{22} - r_{33} = 4q_x^2$ ← $\pm$

$1 - r_{11} + r_{22} - r_{33} = 4q_y^2$

$1 - r_{11} - r_{22} + r_{33} = 4q_z^2$

For highest accuracy solve for the largest term (either sign) then use to solve for the others

$r_{32} - r_{23} = 4q_0 q_x$ ← q_0

$r_{13} - r_{31} = 4q_0 q_y$

$r_{21} - r_{12} = 4q_0 q_z$

$r_{21} + r_{12} = 4q_x q_y$ ← q_y

$r_{32} + r_{23} = 4q_y q_z$

$r_{13} + r_{31} = 4q_z q_x$ ← q_z

eg q_0 is largest.
$q_x = r_{32} - r_{23} / 4q_0$
$q_y = r_{13} - r_{31} / 4q_0$

System II to Quaternion conversion.

$$\begin{bmatrix} q_0^2+q_x^2-q_y^2-q_z^2 & 2(q_yq_x-q_0q_z) & 2(q_zq_x+q_0q_y) \\ 2(q_xq_y+q_0q_z) & q_0^2-q_x^2+q_y^2-q_z^2 & 2(q_zq_y-q_0q_x) \\ 2(q_xq_z-q_0q_y) & 2(q_yq_z+q_0q_x) & (q_0^2-q_x^2-q_y^2+q_z^2) \end{bmatrix} = \begin{bmatrix} c\phi c\theta c\psi - s\phi s\psi & -c\phi c\theta s\psi - s\phi c\psi & c\phi s\theta \\ s\phi c\theta c\psi + c\phi s\psi & -s\phi c\theta s\psi + c\phi c\psi & s\phi s\theta \\ -s\theta c\psi & s\theta s\psi & c\theta \end{bmatrix}$$

$$4q_0^2 = 1 + r_{11} + r_{22} + r_{33} = 1 + c\phi c\theta c\psi - s\phi s\psi + (-s\phi c\theta s\psi + c\phi c\psi) + c\theta$$
$$= 1 + c\theta + c\theta(c\phi c\psi - s\phi s\psi) + (c\phi c\psi - s\phi s\psi) = (1+c\theta) + (1+c\theta)\,c(\phi+\psi)$$
$$4q_0^2 = (1+c\theta)(1+c(\phi+\psi))$$
$$q_0 = \sqrt{\frac{1+c\theta}{2}}\sqrt{\frac{1+c(\phi+\psi)}{2}} = \boxed{+\cos\frac{\theta}{2}\cos\left(\frac{\phi+\psi}{2}\right) = q_0}$$

$$4q_x^2 = 1 + r_{11} - r_{22} - r_{33} = 1 + c\phi c\theta c\psi - s\phi s\psi - (-s\phi c\theta s\psi + c\phi c\psi) - c\theta$$
$$= 1 - c\theta + c\theta(c\phi c\psi + s\phi s\psi) - (c\phi c\psi + s\phi s\psi) = (1-c\theta) - (1-c\theta)\,c(\phi-\psi)$$
$$= (1-c\theta)(1 - c(\phi-\psi))$$
$$q_x = \sqrt{\frac{1-c\theta}{2}}\sqrt{\frac{1-c(\phi-\psi)}{2}} = \boxed{-\sin\frac{\theta}{2}\sin\left(\frac{\phi-\psi}{2}\right) = q_x}$$

$$4q_y^2 = 1 - r_{11} + r_{22} - r_{33} = 1 - (c\phi c\theta c\psi - s\phi s\psi) + -s\phi c\theta s\psi + c\phi c\psi - c\theta$$
$$= (1-c\theta) - c\theta(c\phi c\psi + s\phi s\psi) + (c\phi c\psi + s\phi s\psi) = (1-c\theta) + (1-c\theta)\,c(\phi-\psi)$$
$$= (1-c\theta)(1 + c(\phi-\psi))$$
$$q_y = \sqrt{\frac{1-c\theta}{2}}\sqrt{\frac{1+c(\phi-\psi)}{2}} = \boxed{+\sin\frac{\theta}{2}\cos\left(\frac{\phi-\psi}{2}\right) = q_y}$$

$$4q_z^2 = 1 - r_{11} - r_{22} + r_{33} = 1 - (c\phi c\theta c\psi - s\phi s\psi) - (-s\phi c\theta s\psi + c\phi c\psi) + c\theta$$
$$= (1+c\theta) - c\theta(c\phi c\psi - s\phi s\psi) - (c\phi c\psi - s\phi s\psi) = (1+c\theta) - (1+c\theta)\,c(\phi+\psi)$$
$$= (1+c\theta)(1 - c(\phi+\psi))$$
$$q_z = \sqrt{\frac{1+c\theta}{2}}\sqrt{\frac{1-c(\phi+\psi)}{2}} = \boxed{+\cos\frac{\theta}{2}\sin\left(\frac{\phi+\psi}{2}\right) = q_z}$$

Note: $-(q_0\ q_x\ q_y\ q_z)$ is also a solution

Note: signs were chosen so the following also hold

$$r_{32} - r_{23} = 4q_0q_x$$
$$r_{13} - r_{31} = 4q_0q_y$$
$$r_{21} - r_{12} = 4q_0q_z$$
$$r_{21} + r_{12} = 4q_xq_y$$
$$r_{32} + r_{23} = 4q_yq_z$$
$$r_{13} + r_{31} = 4q_zq_x$$

Quaternion to System II conversion

tem II to Quaternion

$$\boxed{\begin{aligned} q_0 &= \cos\tfrac{\theta}{2}\,\cos\left(\tfrac{\phi+\psi}{2}\right) \\ q_x &= -\sin\tfrac{\theta}{2}\,\sin\left(\tfrac{\phi-\psi}{2}\right) \\ q_y &= \sin\theta/2\,\cos\left(\tfrac{\phi-\psi}{2}\right) \\ q_z &= \cos\theta/2\,\sin\left(\tfrac{\phi+\psi}{2}\right) \end{aligned}}$$

(normal)

$\theta = \cos^{-1}\left(q_0^2 - q_x^2 - q_y^2 + q_z^2\right)$ choose (for now) $0 \le \theta \le 180$

note $\cos\frac{\theta}{2}$ and $\sin\frac{\theta}{2}$ are both positive

If $(q_0 == 0\ \&\&\ q_z == 0)$ sum = 0;
else sum $= \frac{\phi+\psi}{2} = \operatorname{atan2}(q_z, q_0)$ } 180° rotation about an axis in the x-y plane

If $(q_x == 0\ \&\&\ q_y == 0)$ DIFF = 0;
DIFF $= \frac{\phi-\psi}{2} = \operatorname{atan2}(-q_x, q_y)$ } Any rotation about the z-axis

$\phi =$ sum + DIFF

$\psi =$ sum − DIFF

The second solution is given by

$\theta \rightarrow -\theta$ ie $-180 \le \theta \le 0$

$\phi \rightarrow \phi + 180$

$\psi \rightarrow \psi + 180$