Arbitrary Waveform Generators and Numerically Controlled Oscillators

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Introduction

Many DSP systems require the generation of sinusoidal or other periodic waveforms. One method of generating these signals involves using an “Arbitrary Waveform Generator” (ARB) in which a digital accumulator is used to generate the address into a lookup table containing stored values of a single cycle of the waveform. By stepping through the table at an adjustable rate, a wide range of digitally programmable frequencies and phases may be generated. If the stored waveform is sinusoidal (very common), the system is called a “Numerically Controlled Oscillator” (NCO). The system is extremely common, both in hardware and in software. Advantages include the possibility of instantaneous changes in the instantaneous frequency while maintaining a continuous phase property in the output, or allowing a programmable time-varying phase. When integrated with a Digital to Analog Converter (DAC) to create an analog output waveform, the system is called a “Direct Digital Synthesizer” (DDS).

In the following, the NCO is described in terms of the discrete-time frequency (in cycles per sample) being generated. If the system is driven by a fixed clock rate of $F_s$ samples per second, the discrete-time frequency $f_0$ corresponds to the continuous-time frequency $f_0 F_s$ (in Hz).

Arbitrary Waveform Generators

Let $g[k]$ be any periodic function with period $N$. In the final implementation, one period of the waveform is stored in an $N$-sample lookup table ($g[k]$ for $k = 0, 1, \ldots N - 1$). To generate samples of the periodic waveform, the index $k$ is just taken modulo-$N$ to access the stored sample value.

In order to generate samples of the waveform $g[k]$ with an adjustable frequency or phase, the table index $k$ is generated using an accumulator. The value of the index at the $n^{th}$ time step is incremented by a programmable step value $s[n]$. For example, if $s[n]$ takes a constant value of $N/32$, each generated output sample represents $1/32^{nd}$ of a cycle, and it takes 32 time steps to complete one full cycle of the output waveform. In this case, the discrete-time frequency of the output signal is $f_i = 1/32 = 0.03125$ cycles/sample. If the accumulator is being clocked at a rate of $F_s = 64$ MHz, the generated output signal would have a frequency of 2 MHz. In general, if one wishes to set the output signal to a discrete-time instantaneous frequency of $f_i$ cycles/sample, the value of $s[n]$ should be set to $N f_i$.

Most accumulators for waveform generators also allow for a constant offset into the table (which allows the possibility of controlling the phase in addition to the frequency). The table offset at time $n$ is denoted $o[n]$. For example, to introduce a 1/4-cycle phase offset, the value of the table index should be offset by $o[n] = N/4$.

So the “phase accumulator” which generates the index $k$ into the function $g[k]$ includes a step value $s[n]$ which is added to a running phase total, and a constant offset $o[n]$ which is added to the result. The arbitrary waveform generator is illustrated in Figure 1, and the phase accumulator is described by (1) and (2).

$$k_0[n] = k_0[n-1] + s[n] \quad \quad (1)$$
$$k[n] = k_0[n] + o[n] \quad \quad (2)$$

Figure 2 illustrates the generation of a sawtooth waveform using the arbitrary waveform generator.
Figure 1: Arbitrary Waveform Generator Conceptual Structure

Figure 2: Generation of a sawtooth waveform. The illustration shows the generation of a discrete-time frequency of \( f_i = 1/8 \) cycles/sample, with a phase offset of \( 3/8 \) cycles.
Figure 3: Arbitrary Waveform Generator Implementation: M-bit accumulator registers with $2^m$ entries for $g[k]$

**Implementation**

Since the index into the function $g[k]$ is taken modulo-$N$, the system is greatly simplified if the table length $N$ is selected to be an integer power of 2: $N = 2^m$. In this case, an $m$-bit register used for the phase accumulator is simply allowed to overflow. The overflow is ignored, and the $m$ bit register contents are used to form the table index $k$. No explicit modulo operation is needed.

The accuracy of the phase accumulator may be greatly improved by performing the calculation in a register which is larger than $m$ bits, and then using the most-significant $m$ bits of the result to form the index into the lookup table. Assume that $M > m$ bits are used. (For example, $M = 32$ would be a common choice for an architecture with 32-bit registers.) The actual accumulator calculation is obtained by scaling (1) and (2) by a factor of $2^{M-m}$. Define $k'[n]$, $k'_0[n]$, $s'[n]$ and $o'[n]$ by scaling the corresponding terms in (1) and (2) by the factor of $2^{M-m}$. The $M$-bit register calculation is

$$k'_0[n] = k'_0[n-1] + s'[n], \quad k'[n] = k'_0[n] + o'[n] \quad (3)$$

The value of $k[n]$ is obtained from the most significant $m$ bits of $k'[n]$.

The instantaneous frequency and phase offset in terms of the new quantities are

**Instantaneous Frequency**

$$\text{Instantaneous Frequency} = \frac{s[n]}{N} = \frac{s[n]}{2^m} = \frac{s'[n]}{2^M} \text{ cycles/sample} \quad (4)$$

**Phase Offset**

$$\text{Phase Offset} = \frac{o[n]}{N} = \frac{o[n]}{2^m} = \frac{o'[n]}{2^M} \text{ cycles} \quad (5)$$

As a result of the increased word size for the phase accumulator, frequencies and phase offsets may specified with an accuracy of one part in $2^M$.

Figure 3 shows the arbitrary waveform generator in which the phase accumulator uses a larger $M$-bit register size, and an $m$-bit table lookup is used.
The Numerically Controlled Oscillator (NCO)

The NCO is a special case of the arbitrary waveform generator in which the output waveform is a cosine function. In this case the table entries \( g[k] \) are given by

\[
g[k] = \cos(2\pi k/N) \quad k = 0, 1, 2, \ldots, N - 1 \quad N = 2^m
\]

(6)

The frequency and phase relationships for the NCO are the same as those for the arbitrary waveform generator:

\[
\text{Instantaneous Frequency} = s'[n]/2^M \text{ cycles/sample} \quad (7)
\]

\[
\text{Phase Offset} = o'[n]/2^M \text{ cycles} = \frac{2\pi}{2^M} o'[n] \text{ rad.} \quad (8)
\]

Summary relationships for the NCO

- Phase accumulation is performed in an \( M \)-bit integer register. Unsigned operations that “wrap” on overflow are used. Often, \( M \) is selected to to be the natural register size for the processor.

- For an instantaneous input frequency of \( f_i \), the appropriate phase accumulator frequency control word \( s'[n] \) is an integer closest to \( 2^M f_i \). The generated output frequency is \( F_s f_i \) Hz, where \( F_s \) is the sample rate driving the accumulator.

- For a phase-shift of \( \theta_i \) radians, the appropriate phase accumulator offset constant \( o'[n] \) is an integer closest to \( 2^M (\theta_i/2\pi) \).

- The most significant \( m \) bits of the phase accumulator output provide the index into the lookup-table.

- The lookup-table contains samples of exactly one cycle of the desired output waveform.

\[
g[k] = \cos(2\pi k/2^m), \quad k = 0, 1, \ldots, 2^m - 1.
\]

Accuracy

The NCO accuracy is fundamentally limited by the phase accumulator register size (\( M \) bits), the lookup table size (\( 2^m \) entries), and the accuracy of the clock that drives the system.

Large phase accumulator registers (e.g. \( M = 32 \) bits) imply very accurate specification of the instantaneous frequency. For \( M = 32 \), input frequencies are accurate to one part in \( 2^{32} \). This result far exceeds the accuracy of most clock sources, and the accuracy of the generated waveforms tend to be dominated by the accuracy of \( F_s \) or by the size of the lookup table.

Selection of the lookup-table size (\( 2^m \)) is generally limited by the availability of memory. Some savings are possible by going to more elaborate table access techniques (for example, it may be only necessary to store one-fourth of the cosine table, since the remaining values are redundant.) Truncation of the phase word for access to the table causes phase modulation of the output waveform that can generate non-harmonic distortion of the output. Total distortion power can be approximated by examining the worst-case accuracy of NCO output value. If only \( N = 2^m \) values are stored in the table, then the instantaneous phase used to access the table is being rounded to \( 1/N \) cycles, or \( 2\pi/N \) radians. The maximum error occurs when the slope of the cosine is largest (at the cosine zero-crossings). At the zero-crossing, \( \cos() \) has a slope of 1, so the maximum step in the value of \( y(k) \) stored in the table is \( 2\pi/N \). The maximum error in the output sample value is half of this step-size:

\[
E_{max} = \frac{\pi}{N} = \frac{\pi}{2^m}
\]

For example, to achieve a maximum output error of less than 1%, we must require \( N \geq 512 \). Using \( N = 512 \) table entries gives a worst-case amplitude error of about 0.6%. In this case the total distortion power produced by the NCO will be more than 44 dB below the strength of the generated output. Depending upon the requested frequency, this power could be broadly distributed over the spectrum of the \( y(n) \), or may be concentrated in a few discrete frequencies.

Beyond increasing the size of the lookup-table, there are several techniques that can significantly reduce the distortion power. Some systems apply “dithering” to randomize the output of the phase accumulator. While this approach will actually slightly increase the total error output power, it does make the error less correlated with the generated output—the error signal appears more like random noise. Other systems access multiple table entries and use an interpolation process to reduce memory requirements.