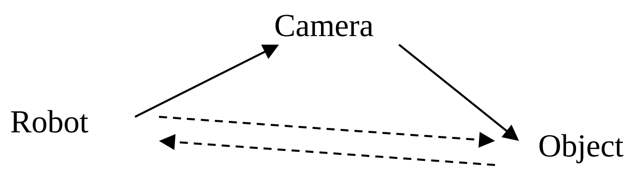
Coordinate Transforms

Robots work in a 3-D world. We need the mathematics for doing this. This involves both position (three degrees of freedom - dof) and orientation (3 dof).

We use the term “pose” refer to position and orientation together (6 dof).

Example:We sense the pose of an object with respect to (wrt) a camera and we know the camera’s pose wrt a robot, so what is the object’s pose wrt the robot? What is the robot’s pose wrt the object?



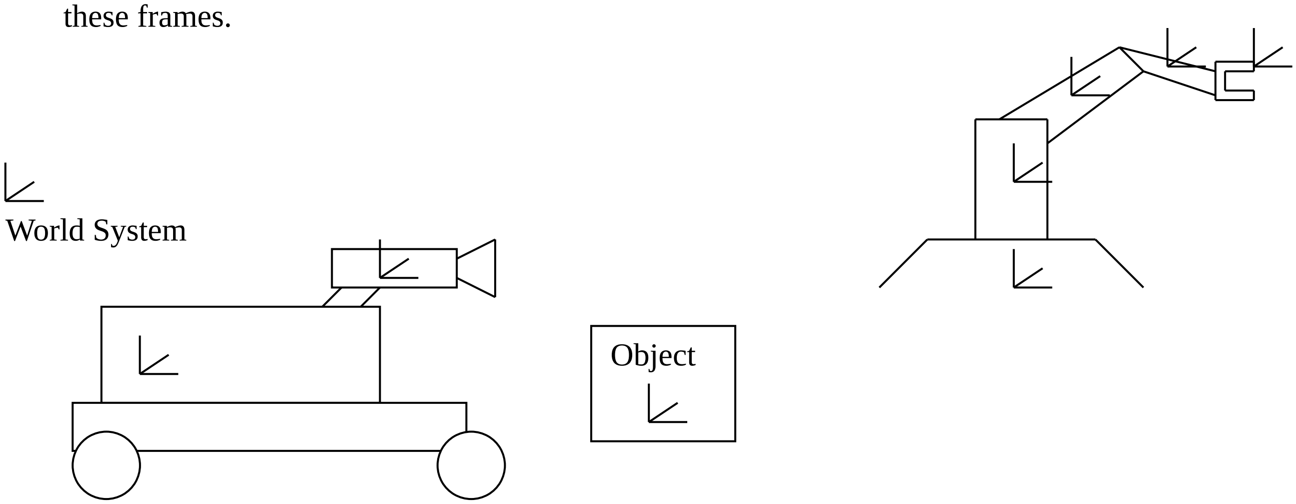
Example: We know the joint angles or robot links and the link geometry. Where is the robot end-effector wrt the robot base?

What we want to know:

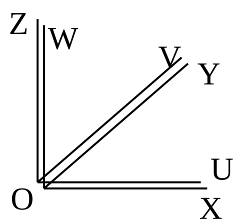
We would like to be able to answer the following four questions:

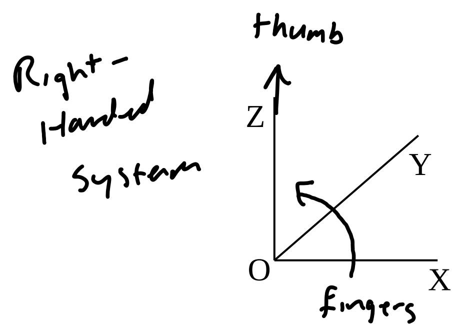
1. How do we represent the pose of **A** wrt **B** (^BT_A)? (Memorize this notation!) this w.r.t. that
2. If we know ^BT_A and ^CT_B, then what is ^CT_A?
3. Given ^BT_A, what is ^AT_B? Inverse
4. If we know the position of points wrt system **A** (^AP) and we know ^BT_A, then what is ^BP?

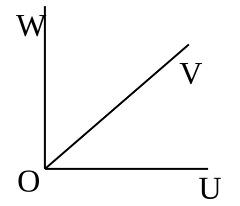
Attach a coordinate frame (system) to each rigid object (or link). Describe the pose by these frames.



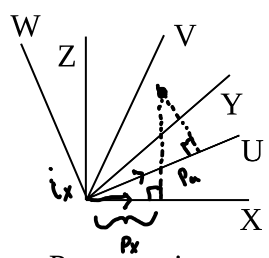
Sometimes we think of the relation between two frames as a sequence of motion to get from one pose to the other. In this case the frames OXYZ (fixed reference system) and OUVW (moving system) are initially aligned:



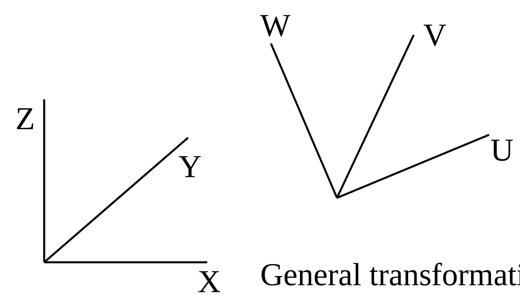




Pure translation



Pure rotation



General transformation

Note that we almost always work with **<u>Right Handed Systems</u>**

Note that bold face letters represent vectors (lower case letters) and matrices (upper case letters). When hand written, we will put a bar over the letter. Scalars are non-bold.

$$\bar{P}_{xyz} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

Transformations

Point **p** can be described wrt either system (basis vectors)

$\mathbf{p}_{XYZ} = (p_X, p_Y, p_Z)^T$ and $\mathbf{p}_{UVW} = (p_U, p_V, p_W)^T$. (Note that $(\cdot)^T$ represents transpose.)

Keep in mind that this is the same point, just represented in two different systems (two sets of “basis vectors”).

What we want to know is

Given $\mathbf{p}_{UVW}$ what is $\mathbf{p}_{XYZ}$?

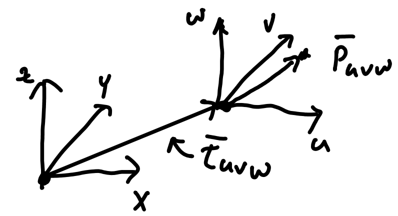
If we can do this in general for all points, then the transformation can be used to represent the pose of OUVW wrt OXYZ.

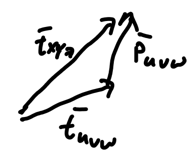
Pure translations:

origin of ouvw wrt. oxyz

$p_X = p_U + t_U$
$p_Y = p_V + t_V$
$p_Z = p_W + t_W$

${}^{xyz}T_{uvw}$





or in vector form

$$\mathbf{p}_{XYZ} = \mathbf{p}_{UVW} + \mathbf{t}_{UVW}$$

The physical meaning of $\mathbf{t}_{UVW} = (t_X, t_Y, t_Z)^T$ is the origin of the OUVW system wrt the OXYZ system.

We could use this system of equations to represent the position of OUVW wrt OXYZ, or more simply we could just use $\mathbf{t}_{UVW}$.

Inverse:

By rewriting the equation to get $\mathbf{p}_{UVW}$ in terms of $\mathbf{p}_{XYZ}$:

$$\mathbf{p}_{UVW} = \mathbf{p}_{XYZ} + (-\mathbf{t}_{UVW})$$

we have an equation which converts points from the OXYZ system to the OUVW system. Obviously, $-\mathbf{t}_{UVW}$ is the *inverse* of $\mathbf{t}_{UVW}$.

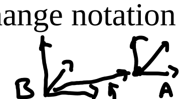
Sequence of Motion:

If you know ${}^{B}T_{A}$ and ${}^{C}T_{B}$, then what is ${}^{C}T_{A}$?

(Note that ${}^{B}T_{A}$ represents a transformation, not a vector or matrix or any particular representation.)

We'll change notation slightly and use ${}^{B}\mathbf{p}$ to represent the point in system **B**, etc. so

Given:



${}^{B}\mathbf{p} = {}^{A}\mathbf{p} + {}^{B}\mathbf{t}_{A}$ ${}^{C}\mathbf{p} = {}^{B}\mathbf{p} + {}^{C}\mathbf{t}_{B}$

Find:
${}^{C}\mathbf{p} = {}^{A}\mathbf{p} + {}^{C}\mathbf{t}_{A}$

Substitute ${}^{B}\mathbf{p}$ from the first equation into the second equation and get:

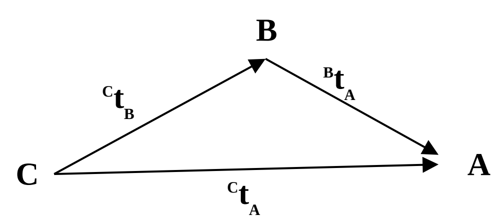
$${}^{C}\mathbf{p} = {}^{A}\mathbf{p} + ({}^{B}\mathbf{t}_{A} + {}^{C}\mathbf{t}_{B})$$

${}^{B}p$

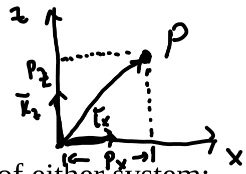
And we can immediately see that

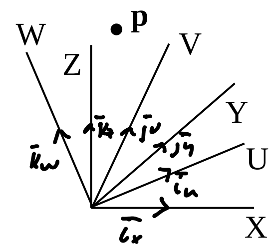
$${}^{C}\mathbf{t}_{A} = {}^{B}\mathbf{t}_{A} + {}^{C}\mathbf{t}_{B}$$

Of course this is just simple vector addition.



Pure Rotation:





Write the same point **p** in terms of either system:

$$\mathbf{p} = p_X\, \mathbf{i}_X + p_Y\, \mathbf{j}_Y + p_Z\, \mathbf{k}_Z$$
$$\mathbf{p} = p_U\, \mathbf{i}_U + p_V\, \mathbf{j}_V + p_W\, \mathbf{k}_W$$

where $\mathbf{i}_X, \mathbf{j}_Y, \mathbf{k}_Z$ and $\mathbf{i}_U, \mathbf{j}_V, \mathbf{k}_W$ are basis vectors in their respective systems and $(p_X, p_Y, p_Z)^T$ and $(p_U, p_V, p_W)^T$ are the coordinates of **p** (i.e., the same point) as described in each of the two systems.

Using a "dot" for dot product note that

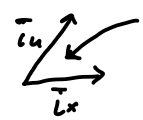
$$p_X = \mathbf{i}_X \cdot \mathbf{p} = \mathbf{i}_X \cdot (p_U\, \mathbf{i}_U + p_V\, \mathbf{j}_V + p_W\, \mathbf{k}_W) = p_U\, \mathbf{i}_X \cdot \mathbf{i}_U + p_V\, \mathbf{i}_X \cdot \mathbf{j}_V + p_W\, \mathbf{i}_X \cdot \mathbf{k}_W$$
$$p_Y = \mathbf{j}_Y \cdot \mathbf{p} = \mathbf{j}_Y \cdot (p_U\, \mathbf{i}_U + p_V\, \mathbf{j}_V + p_W\, \mathbf{k}_W) = p_U\, \mathbf{j}_Y \cdot \mathbf{i}_U + p_V\, \mathbf{j}_Y \cdot \mathbf{j}_V + p_W\, \mathbf{j}_Y \cdot \mathbf{k}_W$$
$$p_Z = \mathbf{k}_Z \cdot \mathbf{p} = \mathbf{k}_Z \cdot (p_U\, \mathbf{i}_U + p_V\, \mathbf{j}_V + p_W\, \mathbf{k}_W) = p_U\, \mathbf{k}_Z \cdot \mathbf{i}_U + p_V\, \mathbf{k}_Z \cdot \mathbf{j}_V + p_W\, \mathbf{k}_Z \cdot \mathbf{k}_W$$

or in matrix form:

cosine angle between

$$\begin{bmatrix} p_X \\ p_Y \\ p_Z \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{i}_X \cdot \mathbf{i}_U & \mathbf{i}_X \cdot \mathbf{j}_V & \mathbf{i}_X \cdot \mathbf{k}_W \\ \mathbf{j}_Y \cdot \mathbf{i}_U & \mathbf{j}_Y \cdot \mathbf{j}_V & \mathbf{j}_Y \cdot \mathbf{k}_W \\ \mathbf{k}_Z \cdot \mathbf{i}_U & \mathbf{k}_Z \cdot \mathbf{j}_V & \mathbf{k}_Z \cdot \mathbf{k}_W \end{bmatrix}}_{{}^{xyz}\bar{R}_{uvw} \quad 3\times 3 \quad \bar{R}} \cdot \underbrace{\begin{bmatrix} p_U \\ p_V \\ p_W \end{bmatrix}}_{3\times 1}$$

Row col



$\bar{i}_x \cdot \bar{i}_u$ = cosine of the angle between

The 3x3 matrix is called a "*direction cosine*" matrix. It gives the cosines of the angles between each axis.

Rotation matrix

Note that this equation transforms points from the OUVW system to the OXYZ systems. We can use the equation (or just the direction cosine matrix) as our representation of the transformation.

In simpler form

$$\underset{3\times 1}{\mathbf{p}_{XYZ}} = \underset{3\times 3}{{}^{XYZ}\mathbf{R}_{UVW}}\, \underset{3\times 1}{\mathbf{p}_{UVW}} = \mathbf{R}\, \mathbf{p}_{UVW}$$

where for the moment we are just using **R** for ${}^{XYZ}\mathbf{R}_{UVW}$.

Inverse:

rewrite the above equation again in terms of $\mathbf{p}_{XYZ}$

$$p_U = \mathbf{i}_U \cdot \mathbf{p} = \mathbf{i}_U \cdot (p_X\, \mathbf{i}_X + p_Y\, \mathbf{j}_Y + p_Z\, \mathbf{k}_Z) = p_X\, \mathbf{i}_U \cdot \mathbf{i}_X + p_Y\, \mathbf{i}_U \cdot \mathbf{j}_Y + p_Z\, \mathbf{i}_U \cdot \mathbf{k}_Z$$
$$p_V = \mathbf{j}_V \cdot \mathbf{p} = \mathbf{j}_V \cdot (p_X\, \mathbf{i}_X + p_Y\, \mathbf{j}_Y + p_Z\, \mathbf{k}_Z) = p_X\, \mathbf{j}_V \cdot \mathbf{i}_X + p_Y\, \mathbf{j}_V \cdot \mathbf{j}_Y + p_Z\, \mathbf{j}_V \cdot \mathbf{k}_Z$$
$$p_W = \mathbf{k}_W \cdot \mathbf{p} = \mathbf{k}_W \cdot (p_X\, \mathbf{i}_X + p_Y\, \mathbf{j}_Y + p_Z\, \mathbf{k}_Z) = p_X\, \mathbf{k}_W \cdot \mathbf{i}_X + p_Y\, \mathbf{k}_W \cdot \mathbf{j}_Y + p_Z\, \mathbf{k}_W \cdot \mathbf{k}_Z$$

or in matrix form:

$$\begin{bmatrix} p_U \\ p_V \\ p_W \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{i}_U \cdot \mathbf{i}_X & \mathbf{i}_U \cdot \mathbf{j}_Y & \mathbf{i}_U \cdot \mathbf{k}_Z \\ \mathbf{j}_V \cdot \mathbf{i}_X & \mathbf{j}_V \cdot \mathbf{j}_Y & \mathbf{j}_V \cdot \mathbf{k}_Z \\ \mathbf{k}_W \cdot \mathbf{i}_X & \mathbf{k}_W \cdot \mathbf{j}_Y & \mathbf{k}_W \cdot \mathbf{k}_Z \end{bmatrix}}_{\bar{R}^T} \cdot \begin{bmatrix} p_X \\ p_Y \\ p_Z \end{bmatrix}$$

$$\bar{P}_{uvw} = \bar{R}^T \bar{P}_{xyz} = \underbrace{\bar{R}^T \bar{R}}_{I} \bar{P}_{uvw}$$

$$\bar{R}^T \bar{R} = I \qquad \bar{R}^{-1} \bar{R} = I \implies \bar{R}^T = \bar{R}^{-1}$$

$$\bar{R} \bar{R}^T = I$$

Which can be written as

$$\mathbf{p}_{UVW} = {}^{UVW}\mathbf{R}_{XYZ}\, \mathbf{p}_{XYZ} = \mathbf{Q}\, \mathbf{p}_{UVW}$$

where for the moment we are just using $\mathbf{Q}$ for ${}^{UVW}\mathbf{R}_{XYZ}$.

By inspection we see that $\mathbf{Q}$ is equal to $\mathbf{R}^T$. We also know that $\mathbf{Q}$ is equal to $\mathbf{R}^{-1}$. This means that $\mathbf{R}$ is an orthonormal matrix (as is $\mathbf{Q}$); i.e., it's inverse is the same as it's transpose.

$\bar{R}^{-1} = \bar{R}^T$

That is:

$$\mathbf{R}\,\mathbf{R}^T = \mathbf{R}\,\mathbf{R}^{-1} = \mathbf{I}$$

This means that the rows of $\mathbf{R}$ are orthogonal unit vectors. This is 6 constraints: the three rows are unit vectors. The three rows are mutually orthogonal. The same can be said of the columns, which gives 6 more equations; however, these 6 new equations are dependent on the first.

Note that $\mathbf{R}$ has nine numbers, which are related by 6 constraints. This means the matrix has 3 degrees of freedom (d.o.f.) to represent the 3 d.o.f for a rotation.

Sequence of motion:

If you know ${}^{B}T_{A}$ and ${}^{C}T_{B}$, then what is ${}^{C}T_{A}$?

(Note that ${}^{B}T_{A}$ represents a transformation, not a vector or matrix or any particular representation.)

Given:

${}^{B}\mathbf{p} = {}^{B}\mathbf{R}_{A}\, {}^{A}\mathbf{p}$

${}^{C}\mathbf{p} = {}^{C}\mathbf{R}_{B}\, {}^{B}\mathbf{p}$

Find:

${}^{C}\mathbf{p} = {}^{C}\mathbf{R}_{A}\, {}^{A}\mathbf{p}$

Substitute ${}^{B}\mathbf{p}$ from the first equation into the second equation and get:

$${}^{C}\mathbf{p} = {}^{C}\mathbf{R}_{B}\, {}^{B}\mathbf{R}_{A}\, {}^{A}\mathbf{p}$$

And we can immediately see that

$${}^{C}\mathbf{R}_{A} = {}^{C}\mathbf{R}_{B}\, {}^{B}\mathbf{R}_{A}$$

rows of R are unit vectors

$$\underbrace{\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}}_{\bar{R}} \underbrace{\begin{pmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{pmatrix}}_{\bar{R}^T} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I} = \begin{array}{l} r_{11}^2 + r_{12}^2 + r_{13}^2 = 1 \\ r_{21}^2 + r_{22}^2 + r_{23}^2 = 1 \\ r_{31}^2 + r_{32}^2 + r_{33}^2 = 1 \\ ① \; r_{11} r_{21} + r_{12} r_{22} + r_{13} r_{23} = 0 \\ ② \; r_{11} r_{31} + r_{12} r_{32} + r_{13} r_{33} = 0 \\ ③ \; r_{21} r_{31} + r_{22} r_{32} + r_{23} r_{33} = 0 \end{array}$$

dot product of rows = 0

orthogonal

$\bar{R}^T = \bar{R}^{-1} \leftrightarrow$ orthonormal ("Proper" Rotation matrix)

Rows of $\bar{R}$ are orthogonal unit vectors ← 6 constraint equations

Cols of $\bar{R}$ are orthogonal unit vectors

Two equivalent sets of equations

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \leftarrow \text{Not orthonormal}$$

Rotation has 3 degrees of freedom

9 numbers in the matrix ← related by 6 constraint equation

Determinant = +1 for RHS → RHS preserving "handed-ness"