Numerically Controlled Oscillators

Don Hummels

1 Introduction

Many DSP systems require the generation of sinusoidal or other periodic waveforms. One method of generating these signals involves "Numerically Controlled Oscillators" (NCOs), in which a digital accumulator is used to generate the address into a sine lookup table. (Other functions could also be stored in the lookup table, creating an "arbitrary waveform generator".) The system is extremely common, both in hardware and in software. It allows instantaneous changes in the instantaneous frequency or phase of the generated waveform, while maintaining a continuous phase property in the output. When integrated with a Digital to Analog Converter (DAC) to create an analog output waveform, the system is called a "Direct Digital Synthesizer" (DDS).

In the following, the NCO is described in terms of the *discrete-time* frequency (in cycles per sample) being generated. If the system is driven by a fixed clock rate of F_s samples per second, the discrete-time frequency f_0 corresponds to the continuous-time frequency $f_0 F_s$ (in Hz).

2 NCO Overview and Derivation

NCOs can be used to generate a wide variety of periodic output waveforms, but for the purposes of this document a cosine output function is assumed.

$$y[n] = \cos(\phi[n]). \tag{1}$$

Here, the argument $\phi[n]$ is the instantaneous phase of the output signal for the n^{th} output sample. For example, for a *fixed* output frequency of f_0 cycles per sample, and a *fixed* output phase of θ_0 radian, $\phi[n] = 2\pi f_0 n + \theta_0$ giving

$$y[n] = \cos(2\pi f_0 n + \theta_0)$$

However, if the frequency and phase are changing with n, the instantaneous frequency $f_i[n]$ provides the *increment* for $\phi[n]$ which must be accumulated over time, while the instantaneous phase offset $\theta_i[n]$ provides an offset value to be applied only to the n^{th} sample.

$$\phi_1[n] = \phi_1[n-1] + 2\pi f_i[n], \qquad \phi[n] = \phi_1[n] + \theta_i[n] \tag{2}$$

An NCO consists of two fundamental blocks: a digital "phase accumulator" to perform the calculation of (2), and a "phase-to-amplitude converter" which converts the values of $\phi(n)$ to create the output sample values given by (1).

The NCO structure is illustrated in Figure 1.

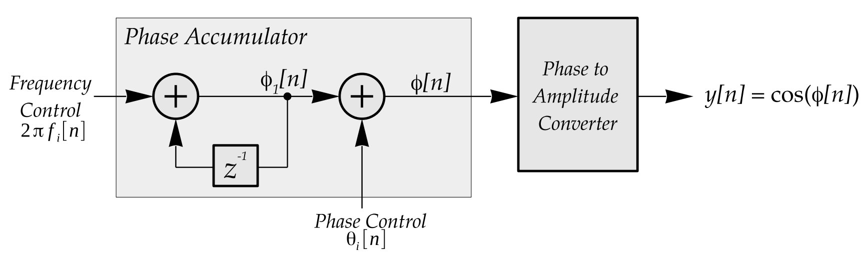

Figure 1: Numerically Controlled Oscillator Conceptual Structure

Phase-to-Amplitude Converter

For efficiency, the Phase-to-Amplitude Converter normally consists of a table of stored pre-calculated values of the $\cos()$ function for phase values encompassing one cycle. An input value of $\phi[n]$ is taken modulo-2π and rounded to find the nearest entry into the table.

Index calculation into the table is greatly simplified if the table length N is selected to be an integer power of 2: $N = 2^m$. In this case, the modulo operation to form the address into the stored table involves simply selecting the appropriate m bits of the phase representation.

Assume that the values stored in the table are given by

$$y_k = \cos(2\pi k/N), \qquad k = 0,\ 1,\ \ldots,\ N-1 \qquad (N = 2^m)$$

Given the desired cosine argument of $\phi[n]$, the index into the table is obtained by setting $\phi[n] = 2\pi k/N$ and solving:

$$k = \frac{N}{2\pi}\phi[n] \tag{3}$$

The resulting index is rounded and taken modulo N.

In practice the calculation in (3) is not used. Instead, equation (2) is scaled by the factor of $N/2\pi$ so that the phase accumulator directly provides the required table index.

Phase Accumulator

The phase accumulator performs the calculation of (2), scaled by a factor of $N/2\pi$ so that the appropriate index into the lookup table is directly generated. Let $k_1[n]$ and $k[n]$ denote the indexes associated with $\phi_1[n]$ and $\phi[n]$ respectively. Scaling (2) by $N/2\pi$ gives the required address calculation:

$$k_1[n] = k_1[n-1] + Nf_i[n], \qquad k[n] = k_1[n] + \frac{N}{2\pi}\theta_i[n] \tag{4}$$

This accumulator is implemented in an integer register that is allowed to "wrap", so that the modulo operation requires no additional hardware or instructions.

The accuracy of the phase accumulator may be greatly improved by performing the calculation in a register which is larger than m bits, and then using the most-significant m bits of the result to form the index into the lookup table. Assume that $M > m$ bits are used. (For example, $M = 32$ would be a common choice for an architecture with 32-bit registers.) The actual accumulator calculation is obtained by scaling (4) by a factor of 2^{M-m}. Define $k'[n] = 2^{M-m}k[n]$ and $k'_1[n] = 2^{M-m}k_1[n]$ as the constants generated in the M-bit register. Scaling (4) by 2^{M-m}, and noting that $2^{M-m}N = 2^{(M-m)}2^m = 2^M$ gives the M-bit register calculation:

$$k'_1[n] = k'_1[n-1] + 2^M f_i[n], \qquad k'[n] = k'_1[n] + \frac{2^M}{2\pi}\theta_i[n] \tag{5}$$

The value of $k[n]$ is obtained from the most significant m bits of $k'[n]$. This value is used to index into the lookup-table. As a result of the increased word size for the phase accumulator, frequencies may specified with an accuracy of one part in 2^M.

3 Summary and Final Structure

The above development is summarized by the following:

- Phase accumulation is performed in an M-bit integer register. Unsigned operations that "wrap" on overflow are used. Often, M is selected to to be the natural register size for the processor.
- For an instantaneous input frequency of f_i, the appropriate phase accumulator frequency control word is an integer closest to $2^M f_i$. The generated output frequency is $F_s f_i$ Hz, where F_s is the sample rate driving the accumulator.
- For a phase-shift of θ_i radians, the appropriate phase accumulator offset constant is an integer closest to $2^M\theta_i/(2\pi)$.
- The most significant m bits of the phase accumulator output provide the index into the lookup-table.

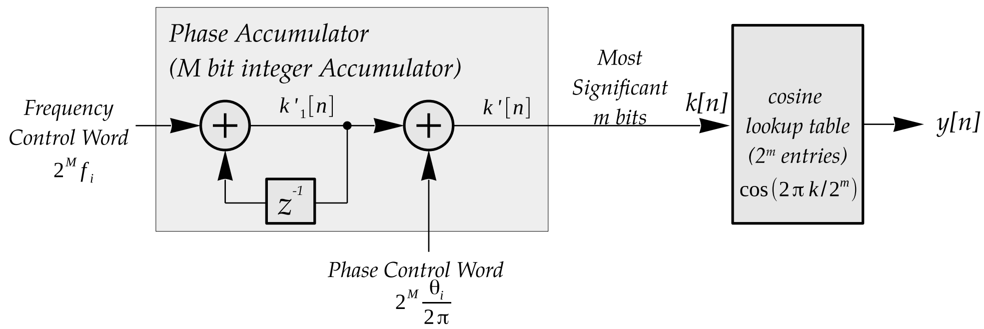

Figure 2: Numerically Controlled Oscillator using integer registers

- The lookup-table contains samples of exactly one cycle of the desired output waveform.

$$y_k = \cos(2\pi k/2^m), \qquad k = 0,\ 1,\ \ldots,\ 2^m - 1.$$

Figure 2 shows the final structure of the NCO.

4 Accuracy

The NCO accuracy is fundamentally limited by the phase accumulator register size (M bits), the lookup table size (2^m entries), and the accuracy of the clock that drives the system.

Large phase accumulator registers (e.g. $M = 32$ bits) imply very accurate specification of the instantaneous frequency. For $M = 32$, input frequencies are accurate to one part in 2^{32}. This result far exceed the accuracy of most clock sources, and the accuracy of the generated waveforms tend to be dominated by the accuracy of F_s or by the size of the lookup table.

Selection of the lookup-table size (2^m) is generally limited by the availability of memory. Some savings are possible by going to more elaborate table access techniques (for example, it may be only necessary to store one-fourth of the cosine table, since the remaining values are redundant.) Truncation of the phase word for access to the table causes phase modulation of the output waveform that can generate non-harmonic distortion of the output. Total distortion power can be approximated by examining the worst-case accuracy of NCO output value. If only $N = 2^m$ values are stored in the table, then the instantaneous phase used to access the table is being rounded to $1/N$ cycles, or $2\pi/N$ radians. The maximum error occurs when the slope of the cosine is largest (at the cosine zero-crossings). At the zero-crossing, $\cos()$ has a slope of 1, so the maximum step in the value of $y(k)$ stored in the table is $2\pi/N$. The maximum error in the output sample value is half of this step-size:

$$E_{max} = \frac{\pi}{N} = \frac{\pi}{2^m}$$

For example, to achieve a maximum output error of less than 1%, we must require $N \geq 512$. Using $N = 512$ table entries gives a worst-case amplitude error of about 0.6%. In this case the total distortion power produced by the NCO will be more than 44 dB below the strength of the generated output. Depending upon the requested frequency, this power could be broadly distributed over the spectrum of the $y(n)$, or may be concentrated in a few discrete frequencies.

Beyond increasing the size of the lookup-table, there are several techniques that can significantly reduce the distortion power. Some systems apply “dithering” to randomize the output of the phase accumulator. While this approach will actually slightly increase the total error output power, it does make the error less correlated with the generated output—the error signal appears more like random noise. Other systems access multiple table entries and use an interpolation process to reduce memory requirements.