Pure rotation
No translation

Rotations about the principal axes.

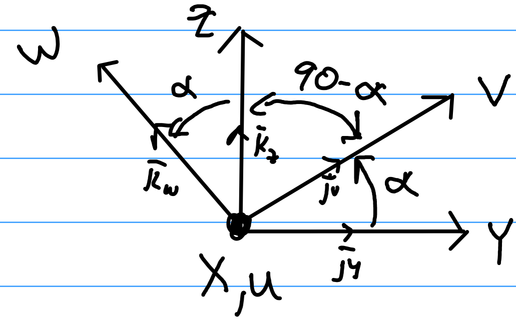
eg.

Rotate about x by α

$$\bar{R}_{x,\alpha} = \begin{bmatrix} \bar{i}_x\cdot\bar{i}_u & \bar{i}_x\cdot\bar{j}_v & \bar{i}_x\cdot\bar{k}_w \\ \bar{j}_y\cdot\bar{i}_u & \bar{j}_y\cdot\bar{j}_v & \bar{j}_y\cdot\bar{k}_w \\ \bar{k}_z\cdot\bar{i}_u & \bar{k}_z\cdot\bar{j}_v & \bar{k}_z\cdot\bar{k}_w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$^{xyz}\bar{R}_{uvw}$

$\cos(90-\alpha)$ $\quad$ $\cos(90+\alpha)$

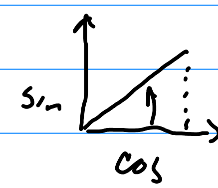


$\cancel{X}$ $\quad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

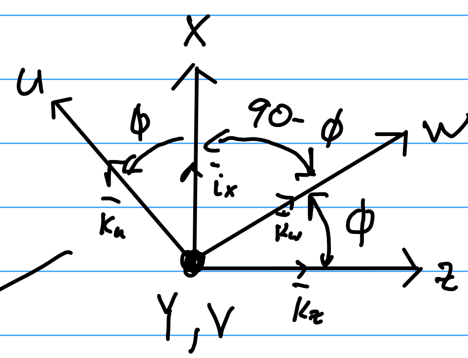
X $\quad \cos(\alpha+\beta) = \cos\alpha\cancel{\cos\beta} - \sin\alpha\overset{+1\;-1}{\sin\beta}$

$-90 \quad -90 \quad -90$

$+90 \quad +90$



$$\begin{bmatrix} \bar{i}_x\cdot\bar{i}_u & \bar{i}_x\cdot\bar{j}_v & \bar{i}_x\cdot\bar{k}_w \\ \bar{j}_y\cdot\bar{i}_u & \bar{j}_y\cdot\bar{j}_v & \bar{j}_y\cdot\bar{k}_w \\ \bar{k}_z\cdot\bar{i}_u & \bar{k}_z\cdot\bar{j}_v & \bar{k}_z\cdot\bar{k}_w \end{bmatrix}$$



$$R_{y,\phi} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{R}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & +\sin\alpha & \cos\alpha \end{bmatrix}$$

$$\bar{R}_{y,\phi} = \begin{bmatrix} \cos\phi & 0 & +\sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \qquad \bar{R}_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ +\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Composite Rotations

$$R_{x,\alpha} \rightarrow \underset{\substack{\uparrow \\ \text{wrt fixed } z}}{R_{z,\theta}} \rightarrow \underset{\substack{\uparrow \\ \text{fixed } Y}}{R_{y,\phi}}$$

$$R_{x,\alpha} \rightarrow \underset{\substack{\uparrow \\ \text{wrt} \\ \text{moving} \\ z}}{R_{w,\theta}} \rightarrow \underset{\substack{\text{wrt} \\ \text{moving} \\ Y}}{R_{v,\phi}}$$

Start with the matrix for the first operation
If motion is w.r.t. OXYZ frame premultiply
If motion is w.r.t. OUVW frame postmultiply

Euler angle representation   3D rotations have 3 d.o.f.

Many different set are possible Three common ones are

System I $R_{z,\phi} \to R_{u,\theta} \to R_{w,\psi} \Rightarrow \bar{R}_{z,\phi}\,\bar{R}_{u,\theta}\,\bar{R}_{w,\psi}$ gyroscopic motion

System II $R_{z,\phi} \to R_{v,\theta} \to R_{w,\psi} \Rightarrow \bar{R}_{z,\phi}\,\bar{R}_{v,\theta}\,\bar{R}_{w,\psi}$ Robotics

System III $R_{x,\psi} \to R_{y,\theta} \to R_{z,\phi} \Rightarrow \bar{R}_{z,\phi}\,\bar{R}_{y,\theta}\,\bar{R}_{x,\psi}$ Roll pitch Yaw Aerospace

System II $\bar{R}_{z,\phi}\,\bar{R}_{v,\theta}\,\bar{R}_{w,\psi}$

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$c\phi = \cos\phi$

$s\phi = \sin\phi$

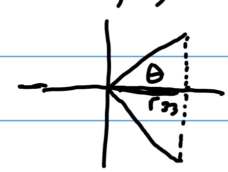
$$\begin{bmatrix} c\phi c\theta & -s\phi & c\phi s\theta \\ s\phi c\theta & c\phi & s\phi s\theta \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

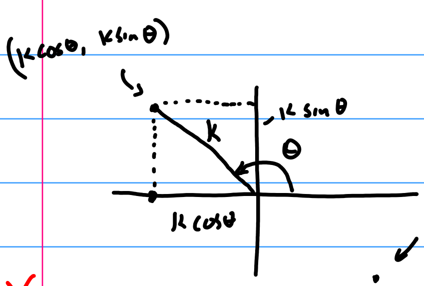
$$\bar{R} \Rightarrow \begin{bmatrix} c\phi c\theta c\psi - s\phi s\psi & -c\phi c\theta s\psi - s\phi c\psi & c\phi s\theta \\ s\phi c\theta c\psi + c\phi s\psi & -s\phi c\theta s\psi + c\phi c\psi & s\phi s\theta \\ \underbrace{-s\theta c\psi}_{r_{31}} & \underbrace{s\theta s\psi}_{r_{32}} & c\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

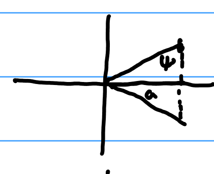
2/18/2021

Given 3x3 Rotation what is ϕ, θ, ψ assuming System II

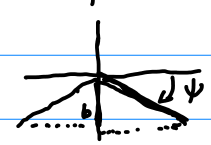
$\theta = \pm\cos^{-1} r_{33}$







$a = \dfrac{r_{31}}{-s\theta}$



$b = \dfrac{r_{32}}{s\theta}$

same atan

X $\theta = \operatorname{atan} y/x = \operatorname{atan}\left(\dfrac{k\sin\theta}{k\cos\theta}\right)$

$\theta = \operatorname{atan2}(y, x) = \operatorname{atan2}(k\sin\theta, k\cos\theta) \quad k > 0$

$= \operatorname{atan2}(-k\sin\theta, -k\cos\theta) \quad k < 0$

System II

$s\theta c\phi = r_{13}$

$s\theta s\phi = r_{23}$

$$\bar{R} \Rightarrow \begin{bmatrix} c\phi c\theta c\psi - s\phi s\psi & -c\phi c\theta s\psi - s\phi c\psi & c\phi s\theta \\ s\phi c\theta c\psi + c\phi s\psi & -s\phi c\theta s\psi + c\phi c\psi & s\phi s\theta \\ -s\theta c\psi & s\theta s\psi & c\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$s\theta c\psi = -r_{31}$

$k\, c\psi = -r_{31}$

$s\theta s\psi = r_{32}$

$k\, s\psi = r_{32}$

$\theta = \pm\cos^{-1} r_{33}$

$\psi = \mathrm{atan2}(r_{32}, -r_{31}) \quad s\theta > 0$

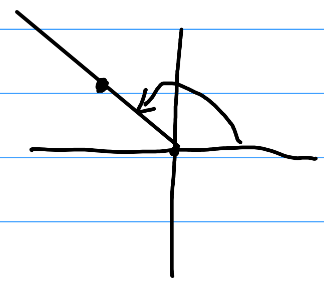
$\quad = \mathrm{atan2}(-r_{32}, r_{31}) \quad s\theta < 0$

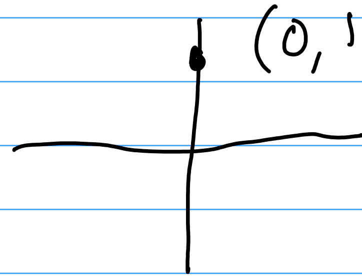
$\phi = \mathrm{atan2}(r_{23}, r_{13}) \quad s\theta > 0$

$\quad \mathrm{atan2}(-r_{23}, -r_{13}) \quad s\theta < 0$

$$\begin{bmatrix} \sim & - & 0 \\ \sim & \sim & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

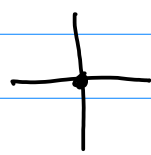
problem #5 HW 4





~~$\mathrm{atan}(y/x)$~~

$\mathrm{atan2}(y, x)$



$\mathrm{atan2}(0, 0) = ?$ arbitrary

$c\phi c\psi - s\phi s\psi = ?\ \cos(\phi + \psi)$

$$\bar{R} \Rightarrow \begin{bmatrix} c\phi \cancel{c\theta}^{\,-1} c\psi - s\phi s\psi & -c\phi \cancel{c\theta}^{\,-1} s\psi - s\phi c\psi & \cancel{c\phi s\theta}\ 0 \\ s\phi \cancel{c\theta}^{\,-1} c\psi + c\phi s\psi & -s\phi \cancel{c\theta}^{\,-1} s\psi + c\phi c\psi & \cancel{s\phi s\theta}\ 0 \\ \cancel{-s\theta c\psi}\ 0 & \cancel{s\theta s\psi}\ 0 & \cancel{c\theta}\ -1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\phi + \psi =$ ___

$R_{z,\phi} \to R_{v,\theta} \to R_{w,\psi}$

40	0	0
20	0	20