Inverse Kinematics

Forward Kinematic plug angles + distances into ${}^{i-1}A_i$
& multiply ${}^{0}A_6 = \prod_{i=1}^{6} {}^{i-1}A_i$

Inverse kinematics is more difficult
The problem is nonlinear
Given 12 values of homogeneous matrix
related by 6 constraint equations
⇒ solve for 6 unknowns

Questions:
① Do solutions exist?
Idea of a workspace
Reachable workspace - can be reached with at least one orientation
Dextrous workspace - can be reached with any orientation
Dextrous is always a subset of reachable

* Workspace may also be limited by joint limits
* If < 6 d.o.f. may only be able to solve for nearest solution
* Tools might have different workspace

② Are there multiple solutions?

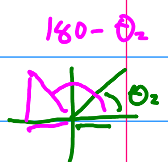
For PUMA-like robot (lasvolt)
① Elbow up - down

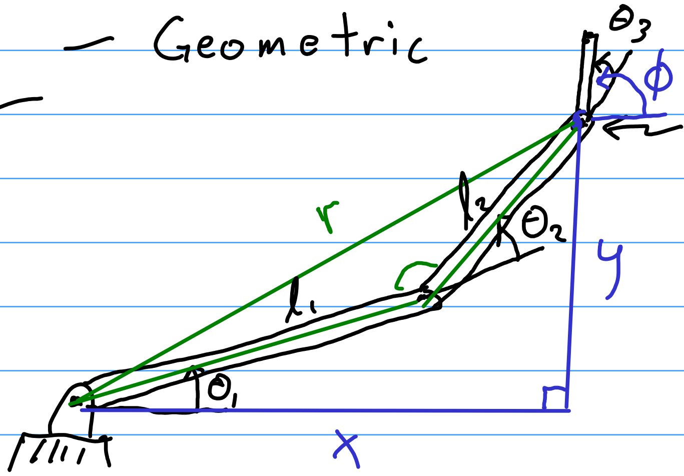
② Arm in front or behind
③ Two solutions with wrist
⇒ 8 solutions

Some choices may exceed joint limits
usually you might choose closest

Solution to inverse kinematics

Method 1 – Geometric





Given (x, y) ϕ

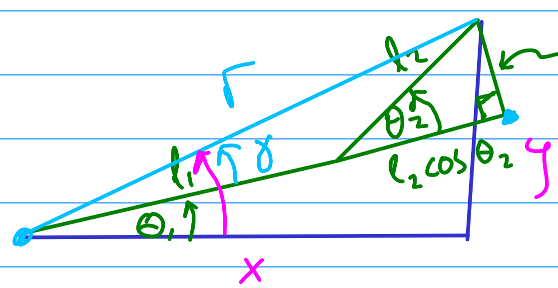
Solve for $\theta_1, \theta_2, \theta_3$

law of cosines

$$r^2 = x^2 + y^2 = l_1^2 + l_2^2 - 2 l_1 l_2 \cos(180^\circ - \theta_2)$$
$$l_1^2 + l_2^2 - 2 l_1 l_2 (-\cos\theta_2)$$
$$l_1^2 + l_2^2 + 2 l_1 l_2 \cos\theta_2$$

$$\cos\theta_2 = \frac{x^2 + y^2 - (l_1^2 + l_2^2)}{2 l_1 l_2}$$

$$\boxed{\theta_2 = \pm \cos^{-1}\left[\frac{x^2 + y^2 - (l_1^2 + l_2^2)}{2 l_1 l_2}\right]}$$



$$\gamma = \operatorname{atan2}(l_2 \sin\theta_2,\ l_1 + l_2 \cos\theta_2)$$
$$\theta_1 + \gamma = \operatorname{atan2}(y, x)$$

$$\gamma = \text{atan2}(l_2 \sin\theta_2,\ l_1 + l_2\cos\theta_2)$$

$$\theta_1 + \gamma = \text{atan2}(y, x)$$

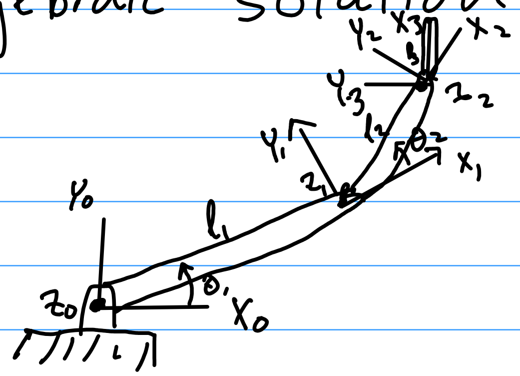
$$\boxed{\theta_1 = \text{atan2}(y, x) - \text{atan2}(l_2 \sin\theta_2,\ l_1 + l_2\cos\theta_2)}$$

$$\phi = \theta_1 + \theta_2 + \theta_3$$

$$\theta_3 = \phi - \theta_1 - \theta_2$$

Method 2

Algebraic solution



	d_i	θ_i	a_i	α_i
1	0	θ_1	l_1	0
2	0	θ_2	l_2	0
3	0	θ_3	0	0

$$^{i-1}T_i = \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & a_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & a_1 C\theta_1 \\ S\theta_1 & C\theta_1 & 0 & a_1 S\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & a_2 C\theta_2 \\ S\theta_2 & C\theta_2 & 0 & a_2 S\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & 0 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & a_1C\theta_1 \\ S\theta_1 & C\theta_1 & 0 & a_1S\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & a_2C\theta_2 \\ S\theta_2 & C\theta_2 & 0 & a_2S\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & 0 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} C\theta_1C\theta_2 - S\theta_1S\theta_2 & -C\theta_1S\theta_2 - S\theta_1C\theta_2 & 0 & a_2C\theta_1C\theta_2 - a_2S\theta_1S\theta_2 + a_1C\theta_1 \\ S\theta_1C\theta_2 + C\theta_1S\theta_2 & -S\theta_1S\theta_2 + C\theta_1C\theta_2 & 0 & a_2S\theta_1C\theta_2 + a_2C\theta_1S\theta_2 + a_1S\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & 0 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} C(\theta_1+\theta_2) & -S(\theta_1+\theta_2) & 0 & a_2C(\theta_1+\theta_2) + a_1C\theta_1 \\ S(\theta_1+\theta_2) & C(\theta_1+\theta_2) & 0 & a_2S(\theta_1+\theta_2) + a_1S\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & 0 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} C(\theta_1+\theta_2)C\theta_3 - S(\theta_1+\theta_2)S\theta_3 & -C(\theta_1+\theta_2)S\theta_3 - S(\theta_1+\theta_2)C\theta_3 & 0 & a_2C(\theta_1+\theta_2) + a_1C\theta_1 \\ S(\theta_1+\theta_2)C\theta_3 + C(\theta_1+\theta_2)S\theta_3 & -S(\theta_1+\theta_2)S\theta_3 + C(\theta_1+\theta_2)C\theta_3 & 0 & a_2S(\theta_1+\theta_2) + a_1S\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_3 = \begin{bmatrix} C(\theta_1+\theta_2+\theta_3) & -S(\theta_1+\theta_2+\theta_3) & 0 & \overset{l_2}{a_2}C(\theta_1+\theta_2) + \overset{l_1}{a_1}C\theta_1 \\ S(\theta_1+\theta_2+\theta_3) & C(\theta_1+\theta_2+\theta_3) & 0 & a_2S(\theta_1+\theta_2) + a_1S\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_3 = \begin{bmatrix} c\phi & -s\phi & 0 & x \\ s\phi & c\phi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\phi = \theta_1 + \theta_2 + \theta_3$$

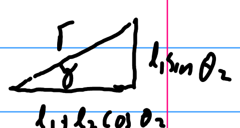
$$\boxed{\theta_3 = \phi - \theta_1 - \theta_2}$$

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

$$x^2 = l_1^2 \cos^2\theta_1 + l_2^2 \cos^2(\theta_1+\theta_2) + 2 l_1 l_2 \cos\theta_1 \cos(\theta_1+\theta_2)$$
$$y^2 = l_1^2 \sin^2\theta_1 + l_2^2 \sin^2(\theta_1+\theta_2) + 2 l_1 l_2 \sin\theta_1 \sin(\theta_1+\theta_2)$$
$$x^2+y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 \cos((\theta_1+\theta_2) - \theta_1)$$

$$x^2+y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 \cos\theta_2$$
$$\boxed{\theta_2 = \pm\cos^{-1}\left(\frac{x^2+y^2-(l_1^2+l_2^2)}{2 l_1 l_2}\right)}$$

$$x = l_1\cos\theta_1 + l_2 \cos\theta_1\cos\theta_2 - l_2 \sin\theta_1 \sin\theta_2$$



$$= \underbrace{(l_1 + l_2\cos\theta_2)}_{r\cos\gamma}\cos\theta_1 - \underbrace{l_2\sin\theta_2}_{r\sin\gamma}\sin\theta_1$$
$$x = r\cos\gamma\cos\theta_1 - r\sin\gamma\sin\theta_1$$
$$\boxed{x = r\cos(\gamma+\theta_1)} \quad \text{where} \quad \boxed{\gamma = \operatorname{atan2}(l_2\sin\theta_2,\ l_1 + l_2\cos\theta_2)}$$

$$y = l_1\sin\theta_1 + l_2\sin\theta_1\cos\theta_2 + l_2\cos\theta_1\sin\theta_2$$
$$= \underbrace{(l_1 + l_2\cos\theta_2)}_{r\cos\gamma}\sin\theta_1 + \underbrace{l_2\sin\theta_2}_{r\sin\gamma}\cos\theta_1$$
$$y = r\cos\gamma\sin\theta_1 + r\sin\gamma\cos\theta_1$$
$$\boxed{y = r\sin(\gamma+\theta_1)}$$

$$\gamma + \theta_1 = \operatorname{atan2}(y, x)$$

$$\boxed{\theta_1 = \operatorname{atan2}(y, x) - \operatorname{atan2}(l_2\sin\theta_2,\ l_1 + l_2\cos\theta_2)}$$

$$\boxed{\theta_3 = \phi - \theta_1 - \theta_2}$$

Top view