Quaternions

To represent an orientation ← 3 dof

① $\tilde{R}_{33}$ ↑ orthonormal, 9 numbers, 6 constraints

② Euler Angles ↑ 3 numbers, system II $R_{z,\phi} \to R_{v,\theta} \to R_{w,\psi}$

$(0, \epsilon, 0)$ ← almost same

$(90, \epsilon, -90)$ ↙

③ Axis-Angle: 4 numbers, 1 constraint

$\epsilon^\circ\ (1, 0, 0)$

$\epsilon^\circ\ (0, 0, 1)$

④ Quaternions: 4-numbers, 1-constraint

Axis-Angle: $R_{\bar{n},\theta}$ θ° rotation about axis $\bar{n} = \begin{pmatrix} n_i \\ n_j \\ n_k \end{pmatrix}$ ← unit vector

Quaternion $\mathring{q} = \left(\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)n_i, \sin\left(\frac{\theta}{2}\right)n_j, \sin\left(\frac{\theta}{2}\right)n_k\right)^T$

$(s, a, b, c)^T$

unit vector

$$s^2+a^2+b^2+c^2 = C^2\tfrac{\theta}{2} + S^2\tfrac{\theta}{2} n_i^2 + S^2\tfrac{\theta}{2} n_j^2 + S^2\tfrac{\theta}{2} n_k^2$$

$$= C^2\tfrac{\theta}{2} + S^2\tfrac{\theta}{2}(n_i^2+n_j^2+n_k^2)$$

$$= C^2\tfrac{\theta}{2} + S^2\tfrac{\theta}{2}$$

$$= 1$$

$\mathring{q} = (s, a, b, c)^T$ ← assign units to different positions

$1 \quad \bar{i} \quad \bar{j} \quad \bar{k}$

$\mathring{q} = s + a\bar{i} + b\bar{j} + c\bar{k}$

like complex (a, b), $c = a + bi$, $i^2 = -1$

$$\bar{i}^2 = \bar{j}^2 = \bar{k}^2 = \bar{i}\bar{j}\bar{k} = -1$$

$\bar{i}\bar{j} = \bar{k} \qquad \bar{j}\cdot\bar{k} = \bar{i} \qquad \bar{k}\bar{i} = \bar{j}$

$\bar{j}\bar{i} = -\bar{k} \qquad \bar{k}\bar{j} = -\bar{i} \qquad \bar{i}\bar{k} = -\bar{j}$

$$\bar{i}^2 = \bar{j}^2 = \bar{k}^2 = \bar{i}\bar{j}\bar{k} = -1$$

$$\bar{i}\bar{j} = \bar{k} \qquad \bar{j}\cdot\bar{k} = i \qquad \bar{k}\bar{i} = \bar{j}$$

$$\bar{j}\bar{i} = -\bar{k} \qquad \bar{k}\bar{j} = -i \qquad \bar{i}\bar{k} = -\bar{j}$$

$$\mathring{q} = s + \underbrace{a\bar{i} + b\bar{j} + c\bar{k}}_{\text{vector part}} = s + \bar{v} \qquad \mathring{q} = \begin{pmatrix} s \\ a \\ b \\ c \end{pmatrix} \rightarrow \mathring{q}^* = \begin{pmatrix} s \\ -a \\ -b \\ -c \end{pmatrix}$$

* For an orientation represented by $\mathring{q}$ its inverse is $\mathring{q}^*$

$$\mathring{q}^* = s - \bar{v} = s - a\bar{i} - b\bar{j} - c\bar{k}$$

Two orientations $\mathring{q}_1$ & $\mathring{q}_2$ $\qquad \bar{R}_1 \cdot \bar{R}_2 = \bar{R}_3$

To concatenate you multiply $\qquad \mathring{q}_1 \mathring{q}_2 = \mathring{q}_3$

$$\mathring{q}_3 = \mathring{q}_1 \mathring{q}_2 = (s_1 + a_1\bar{i} + b_1\bar{j} + c_1\bar{k})(s_2 + a_2\bar{i} + b_2\bar{j} + c_2\bar{k})$$

$$= s_1 s_2 - a_1 a_2 - b_1 b_2 - c_1 c_2 \quad \text{— scalar part}$$

$$+ (s_1 a_2 + a_1 s_2 + b_1 c_2 - c_1 b_2)\,\bar{i}$$

$$+ (s_1 b_2 + b_1 s_2 - a_1 c_2 + c_1 a_2)\,\bar{j}$$

$$+ (s_1 c_2 + c_1 s_2 + a_1 b_2 - b_1 a_2)\,\bar{k}$$

similar to $(a + b_i)(c + d_i)$

$$\mathring{q}_3 = \underbrace{\begin{bmatrix} s_1 & -a_1 & -b_1 & -c_1 \\ a_1 & s_1 & -c_1 & b_1 \\ b_1 & c_1 & s_1 & -a_1 \\ c_1 & -b_1 & a_1 & s_1 \end{bmatrix}}_{Q_1} \underbrace{\begin{pmatrix} s_2 \\ a_2 \\ b_2 \\ c_2 \end{pmatrix}}_{\mathring{q}_2} = \underbrace{\begin{pmatrix} s_3 \\ a_3 \\ b_3 \\ c_3 \end{pmatrix}}_{\mathring{q}_3}$$

$$\mathring{q}_1 \mathring{q}_2 = (s_1 + \bar{v}_1)(s_2 + \bar{v}_2)$$

$$= \underbrace{s_1 s_2 - \bar{v}_1 \cdot \bar{v}_2}_{\text{real part}} + \underbrace{s_2 \bar{v}_1 + s_1 \bar{v}_2 + \bar{v}_1 \times \bar{v}_2}_{\text{vector part}}$$

dot product

cross product