$$
\begin{array}{cccc}
a & 0 & 0 & 1 & 1 \\
+b & +0 & +1 & +0 & +1 \\
\hline
s_1 s_0 & 0\ 0 & 0\ 1 & 0\ 1 & 1\ 0
\end{array}
$$

(a) Evaluation of $S = a + b$

a	b	s_1	s_0
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0
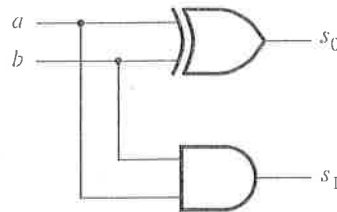
(b) Truth table   (c) Logic network

**Figure 2.12**   Addition of binary numbers.

2.5   BOOLEAN ALGEBRA

In 1849 George Boole published a scheme for the algebraic description of processes involved in logical thought and reasoning [1]. Subsequently, this scheme and its further refinements became known as *Boolean algebra*. It was almost 100 years later that this algebra found application in the engineering sense. In the late 1930s Claude Shannon showed that Boolean algebra provides an effective means of describing circuits built with switches [2]. The algebra can, therefore, be used to describe logic circuits. We will show that this algebra is a powerful tool that can be used for designing and analyzing logic circuits. The reader will come to appreciate that it provides the foundation for much of our modern digital technology.

Axioms of Boolean Algebra

Like any algebra, Boolean algebra is based on a set of rules that are derived from a small number of basic assumptions. These assumptions are called *axioms*. Let us assume that Boolean algebra involves elements that take on one of two values, 0 and 1. Assume that the following axioms are true:
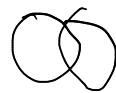
1a. $0 \cdot 0 = 0$
1b. $1 + 1 = 1$
2a. $1 \cdot 1 = 1$

2b. $0 + 0 = 0$
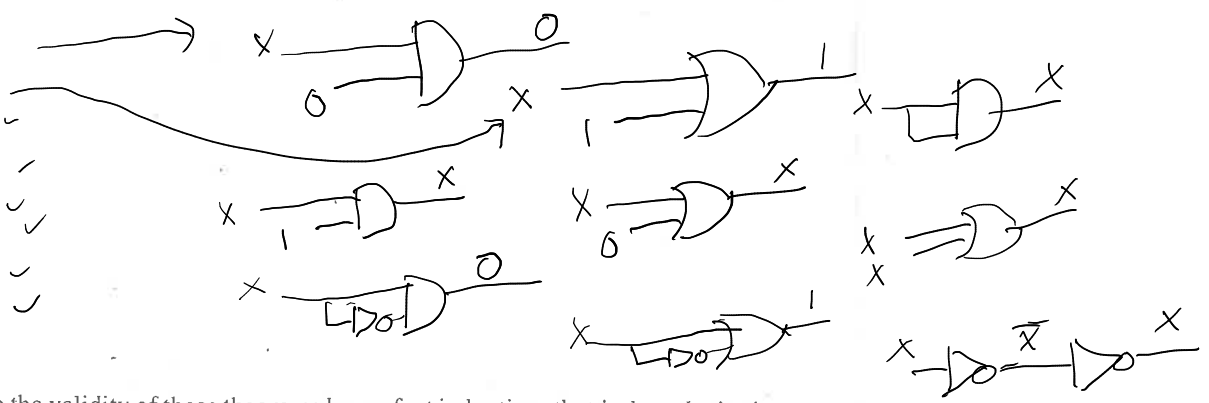
3a. $0 \cdot 1 = 1 \cdot 0 = 0$

3b. $1 + 0 = 0 + 1 = 1$

4a.   If $x = 0$, then $\bar{x} = 1$

4b.   If $x = 1$, then $\bar{x} = 0$

Single-Variable Theorems

From the axioms we can define some rules for dealing with single variables. These rules are often called *theorems*. If x is a Boolean variable, then the following theorems hold:

5a. $x \cdot 0 = 0$

5b. $x + 1 = 1$

6a. $x \cdot 1 = x$

6b. $x + 0 = x$

7a. $x \cdot x = x$

7b. $x + x = x$

8a. $x \cdot \bar{x} = 0$

8b. $x + \bar{x} = 1$

9. $\bar{\bar{x}} = x$

It is easy to prove the validity of these theorems by perfect induction, that is, by substituting the values $x = 0$ and $x = 1$ into the expressions and using the axioms given above. For example, in theorem 5a, if $x = 0$, then the theorem states that $0 \cdot 0 = 0$, which is true according to axiom 1a. Similarly, if $x = 1$, then theorem 5a states that $1 \cdot 0 = 0$, which is also true according to axiom 3a. The reader should verify that theorems 5a to 9 can be proven in this way.

Duality

Notice that we have listed the axioms and the single-variable theorems in pairs. This is done to reflect the important *principle of duality*. Given a logic expression, its *dual* is obtained by replacing all + operators with · operators, and vice versa, and by replacing all 0s with 1s, and vice versa. The dual of any true statement (axiom or theorem) in Boolean algebra is also a true statement. At this point in the discussion, the reader might not appreciate why duality is a useful concept. However, this concept will become clear later in the chapter, when we will show that duality implies that at least two different ways exist to express every logic function with Boolean algebra. Often, one expression leads to a simpler physical implementation than the other and is thus preferable.

Two- and Three-Variable Properties

To enable us to deal with a number of variables, it is useful to define some two- and three-variable algebraic identities. For each identity, its dual version is also given. These identities are often referred to as *properties*. They are known by the names indicated below. If x, y, and z are Boolean variables, then the following properties hold:

10a. $x \cdot y = y \cdot x$ *Commutative*
10b. $x + y = y + x$
11a. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ *Associative*
11b. $x + (y + z) = (x + y) + z$
12a. $x \cdot (y + z) = x \cdot y + x \cdot z$ *Distributive*
12b. $x + y \cdot z = (x + y) \cdot (x + z)$
13a. $x + x \cdot y = x$ *Absorption*
13b. $x \cdot (x + y) = x$
14a. $x \cdot y + x \cdot \bar{y} = x$ *Combining*
14b. $(x + y) \cdot (x + \bar{y}) = x$
15a. $\overline{x \cdot y} = \bar{x} + \bar{y}$ *DeMorgan's theorem*
15b. $\overline{x + y} = \bar{x} \cdot \bar{y}$
16a. $x + \bar{x} \cdot y = x + y$
16b. $x \cdot (\bar{x} + y) = x \cdot y$
17a. $x \cdot y + y \cdot z + \bar{x} \cdot z = x \cdot y + \bar{x} \cdot z$ *Consensus*
17b. $(x + y) \cdot (y + z) \cdot (\bar{x} + z) = (x + y) \cdot (\bar{x} + z)$
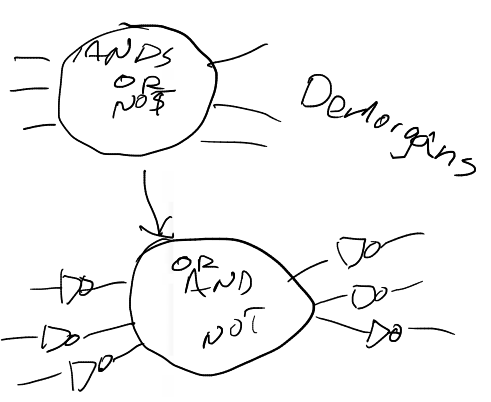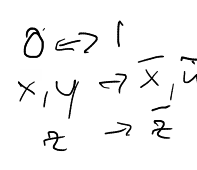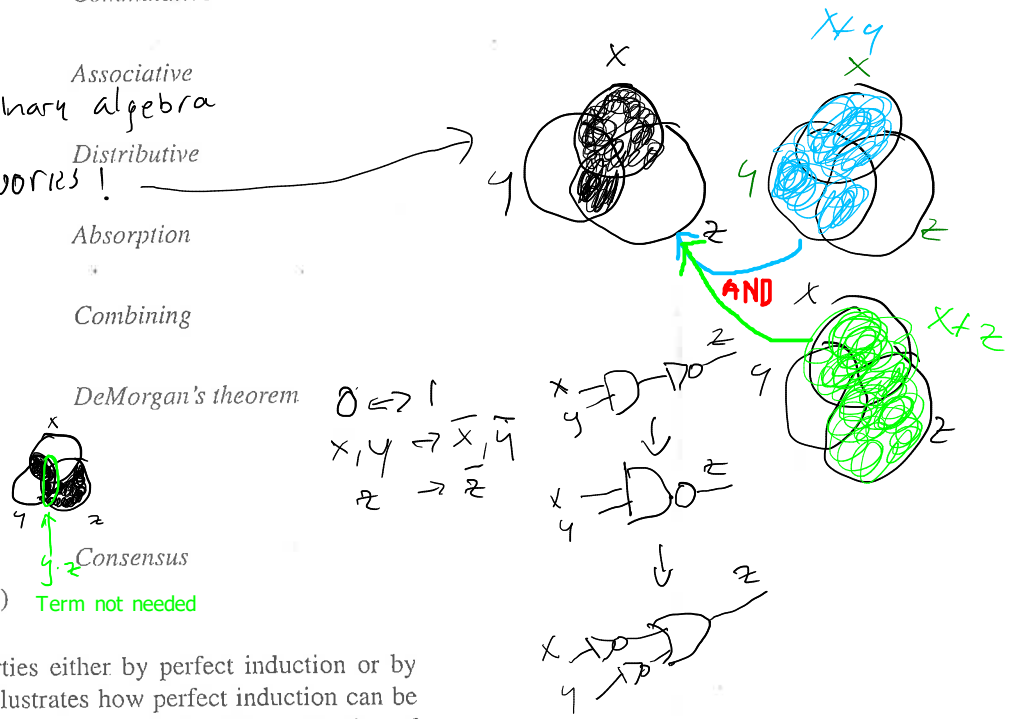
Again, we can prove the validity of these properties either by perfect induction or by performing algebraic manipulation. Figure 2.13 illustrates how perfect induction can be used to prove DeMorgan's theorem, using the format of a truth table. The evaluation of left-hand and right-hand sides of the identity in 15a gives the same result.

We have listed a number of axioms, theorems, and properties. Not all of these are necessary to define Boolean algebra. For example, assuming that the + and · operations are defined, it is sufficient to include theorems 5 and 8 and properties 10 and 12. These are sometimes referred to as Huntington's basic postulates [3]. The other identities can be derived from these postulates.
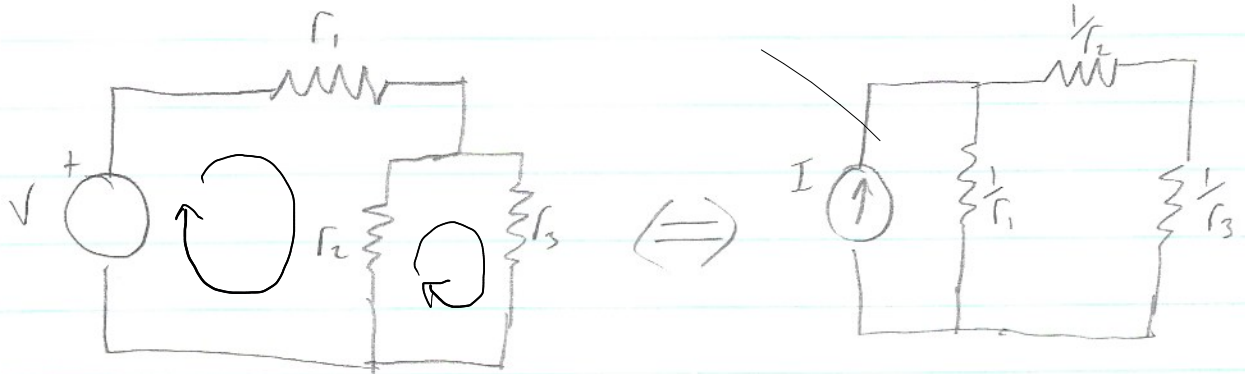
The preceding axioms, theorems, and properties provide the information necessary for performing algebraic manipulation of more complex expressions.

x	y	$x \cdot y$	$\overline{x \cdot y}$	$\bar{x}$	$\bar{y}$	$\bar{x} + \bar{y}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

LHS   RHS

Figure 2.13   Proof of DeMorgan's theorem in 15a.

Duality

$$V \longrightarrow I$$
$$I \longrightarrow V$$
$$r \longrightarrow \frac{1}{r}$$

nodes $\longrightarrow$ loops

loops $\longrightarrow$ nodes